Bifurcation Buckling as an Approximation of the Collapse Load for General Shells

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A computer program, STAGS, for nonlinear analysis of shells of general shape has been extended to include a branch for bifurcation buckling analysis. In the case of general shells, failure usually occurs by means of collapse at a limit point rather than through bifurcation. Therefore, the paper contains also a discussion of the practical applicability of the bifurcation buckling theory. Several example cases are presented in which results from a bifurcation buckling analysis are compared to results from a rigorous nonlinear analysis. It is emphasized by these examples that the classical buckling analysis may give results of little or no value if the shell geometry deteriorates appreciably (Brazier effect) or stresses are redistributed (statical indeterminance) in the subcritical load range. On the other hand, there are cases in which the less costly bifurcation analysis can be substituted for the rigorous collapse analysis.

Introduction

COMPLICATED shell structures for which the weight economy is of utmost importance, such as present space shuttle configurations, have resulted in an increase in the interest in two-dimensional shell analysis. Simultaneously remarkable improvements in numerical methods and in computer technology have greatly enhanced our capability to handle such problems. Hence, computer programs for the buckling analysis of shells of general shape or loading are being offered to a public which to a large extent may be unaware of the limitations of the theory on which these programs are based. A computer program for buckling analysis of shells of general shape or loading based on the classical bifurcation theory must be used with some caution.

A bifurcation point indicates a load level at and above which some new deformation mode is possible. Therefore, the bifurcation analysis is a rigorous solution of the problem only if the failure mode is orthogonal to the prebuckling deformation pattern. This, of course, is unlikely to be the case if the shape of the shell or its loading is of a general nature, and for such shells buckling or collapse will in most cases occur through the passing of a limit point (a maximum in a load displacement curve). However, the "classical" buckling theory (bifurcation from linear membrane solution) sometimes gives good approximations to the buckling load even for cases which are outside its scope. For an axially loaded cylinder with the edges restrained from radial displacements, the rigorous nonlinear analysis shows that the displacements approach infinity as the axial stress approaches the critical value

$$\sigma_{CR} = Et/\{R[3(1 - \nu^2)]^{\frac{1}{2}}\} \tag{1}$$

If in the bifurcation buckling equations, all terms are deleted which contain prebuckling quantities other than membrane stresses, the critical load (for axisymmetric buckling) is found to be in agreement with the rigorous solution [Eq. (1)]. Another example in which the classical buckling analysis gives very good results, although it is not rigorously applicable, is the bending of cylinders which are thin enough to make the Brazier effect negligible. One example in which the classical approach fails to give acceptable results is the axisymmetric buckling of clamped shallow spherical shells under external pressure.

An extension to a computer program for nonlinear analysis (STAGS) is presented here and in view of the preceding discussion, a parameter study is added with the purpose of shedding some light on the practical applicability of the approach.

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The STAGS computer program has been under development for about three years. The program performs a nonlinear analysis of shells by use of a two-dimensional finite difference grid and an energy principle. Displacement and stress histories are computed corresponding to a given history of applied load, displacement or temperature. The theoretical background for STAGS is presented in Ref. 1 and its scope is discussed in more detail in Ref. 2. It applies to any shell for which a reference surface and a suitable set of gridlines (following shell boundaries) can be defined. The shell wall thickness can vary and material properties can vary with the surface coordinates and through the thickness. Cutouts in the shell wall and discrete stiffening can be included.

Analysis

The analysis of bifurcation buckling in the STAGS program is based on energy methods in combination with a two-dimensional finite difference discretization. The shell surface is covered with mesh lines parallel to the coordinate lines; an n-dimensional vector X of unknown displacement components u, v, and w is defined at discrete points on the surface. The nonlinear expressions for the potential energy of deformation of a general shell are given in Ref. 1. After the displacements and their derivatives have been replaced by finite difference approximations, the strain energy density at a mesh station i can be written in the form

$$\Delta U^i = \tfrac{1}{2} Z^{i*} D^i Z^i \tag{2}$$

where D is a 6×6 matrix of constants and Z^i is the vector of strains and curvature changes at station i (all vectors are understood to be column vectors and * designates the adjoint operator: thus, Z^{i*} is a row vector). The matrix D^i is obtained by integration through the shell wall and is a function of the geometric parameters of the shell and of the material properties. The strain vector Z^i is a nonlinear (quadratic) function of the displacement unknowns and the geometric parameters. The vector of stress resultants at station i is given by

$$S^i = D^i Z^i \tag{3}$$

The total strain energy of the shell is then

$$U = \sum_i \Delta U_i a^i \tag{4}$$

where a^i is the area of the ith subregion of the mesh. The work W done by external forces may be expressed in discrete form as

$$W = X^* F \tag{5}$$

where F is the vector of external forces. As the strain expressions are of second order in the displacement components, the total potential energy, V, of the shell is a polynomial of 4th degree in the discrete displacement components, defined by

$$V = U - W \tag{6}$$

A necessary condition for the static equilibrium is that the potential energy be a minimum. For a polynomial, this requires that the gradient of V vanishes leading to the equation

$$LX = F \tag{7}$$

Here, L is defined as the nonlinear operator such that

$$LX = \text{Grad } U \tag{8}$$

The derivation of the complete nonlinear solution of Eq. (7) as well as of bifurcation buckling is facilitated by introduction of the concept of the derivative L' of L.[3] In particular, for the operator L, the derivative L' (sometimes called the Frechet derivative of L) is an $n \times n$ matrix whose elements are

$$L'_{i,j} = \partial^2 U / \partial X_{(i)} \, \partial X_{(j)} \tag{9}$$

Because L' is a function of a particular displacement vector X (unless the nonlinear terms are dropped), the Frechet derivative will usually be denoted L'_X to indicate this dependence. With the use of the derivative L' of the operator L, Newton's method may be readily generalized to obtain the solution of Eq. (7). The iteration is defined by

$$L'^{k}_{X}(X_{k+1} - X_k) = F - LX_k \tag{10}$$

If X_O is an estimate sufficiently close to the solution $\bar{X}$ and if $L'_{\bar{X}}$ is not a singular matrix, the iteration converges to $\bar{X}$. Under these assumptions, it also can be shown that the converged solution is unique.[3]

The application of Newton's method and the modified Newton method in STAGS to obtain a nonlinear collapse analysis is discussed in detail in Ref. 2. It is interesting to note that the mathematical characterization of bifurcation buckling also is provided by the generalized Newton method. Let $\bar{X}$ be a solution of Eq. (7) under a given vector F of external forces. If every neighborhood of $\bar{X}$ contains another vector Y which satisfies the equation

$$LY = F \tag{11}$$

then bifurcation is said to occur for the shell under the load F. From the previous remarks on the conditions for convergence of Newton's method to a unique solution, it follows that a necessary condition for bifurcation is that $L'_{\bar{X}}$ be a singular matrix, or

$$\det(L'_{\bar{X}}) = 0 \tag{12}$$

Classical bifurcation buckling theory may be easily obtained from Eq. (12). It is assumed that $\bar{X}$ may be written

$$\bar{X} = \lambda X_L \tag{13}$$

where X_L is the linear solution for a load vector F_L. Thus, Eq. (12) becomes

$$\det(L'_{\lambda X_L}) = 0 \tag{14}$$

Equation (14) defines an algebraic eigenvalue problem of the form

$$\det(A - \lambda B - \lambda^2 C) = 0 \tag{15}$$

In classical bifurcation theory, the C matrix, which arises from the prebuckling rotations, is often omitted and the eigenvalue problem

$$AX = \lambda BX \tag{16}$$

is obtained.

When bifurcation is considered but the prebuckling displacements are not linear, the solution of Eq. (12) generally requires a stepwise procedure. One such method is given by the recurrence equations

$$\det(L'_{\lambda_{k+1} X_k}) = 0$$
$$X_{k+1} = \lambda_{k+1} X_k \tag{17}$$

in which the starting vector X_O may be represented by the linear solution.

A sequence of eigenvalue problems is solved and, if the method is successful, λ_k approaches one. The bifurcation load factor λ is then represented by $\lambda = \prod_{k=1}^{\infty} \lambda_k$. A nonlinear bifurcation treatment [equivalent to Eq. (17)] was presented in Ref. 6 and has been used successfully to study a large variety of problems. For the two-dimensional problems under consideration here, it appears that such methods may be as costly as the complete nonlinear analysis. Consequently, only a classical bifurcation buckling analysis is implemented in the STAGS program.

The formation of the A and B matrices of Eq. (16) will be considered briefly. The elements of the Frechet derivative matrix L'_X (which define the matrices A and B) are determined according to Eq. (9). The rules for computing derivatives of polynomials are easily programed, and the formation of the A and B matrices, therefore, is well suited to automatic treatment on the computer. Thus, for example, if $X_{(i)}$ and $X_{(j)}$ are the ith and jth displacement components, we have, using Eqs. (2–4),

$$\partial^2 U / \partial X_{(i)} \, \partial X_{(j)} = \sum_{k=1}^{m} a^k (\partial^2 \Delta U^k / \partial X_{(i)} \, \partial X_{(j)}) \tag{18}$$

Examining the kth term of this sum, with $X = \lambda X_L$,

$$\frac{\partial^2 \Delta U^k}{\partial X_{(i)} \, \partial X_{(j)}} = \frac{\partial^2 Z^{k*}}{\partial X_{(i)} \, \partial X_{(j)}} \lambda S^k + \frac{\partial Z^{k*}}{\partial X_{(i)}} D^k \frac{\partial Z^k}{\partial X_{(j)}} \tag{19}$$

In the first term on the right-hand side of Eq. (18), note that S^k is the linear stress resultant vector at station k and that only the quadratic terms (rotations) need be considered in forming the partial derivatives $\partial^2 Z^{k*} / \partial X_{(i)} \, \partial X_{(j)}$. Contributions from this term go into the B matrix. Assuming the prebuckling rotations may be neglected for the classical theory, the last term of Eq. (19) generates additions only to the A matrix. The A matrix is then identical to the linear stiffness matrix. If the prebuckling rotations are included, the last term of Eq. (19) generates a C matrix and provides additional contributions to the B matrix. In this case, the prebuckling stress resultant vector S would also include nonlinear terms.

In conclusion, it should be noted that bifurcation buckling theory is often based on the concept of adjacent equilibrium states. Of course, the same algebraic eigenvalue problem is ultimately obtained by both methods. However, the approach presented here seems to provide a simpler recipe for definition of the basic matrices of Eq. (16). The recipe is outlined in Eqs. (18) and (19) and leads to straightforward algebraic procedures. In addition, the relations between linear and nonlinear bifurcation theory and Newton's method are clarified.

Numerical Results

The difference between the nonlinear behavior and the idealization of "bifurcation behavior" of a shell of general shape is illustrated in Fig. 1. An elliptical cone with the dimensions shown in the figure is subjected to axial compression through the application of a uniform end shortening. The collapse load is

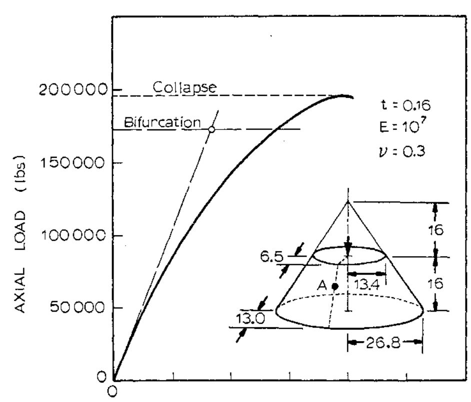

Fig. 1 Elliptic cone under uniform end shortening.

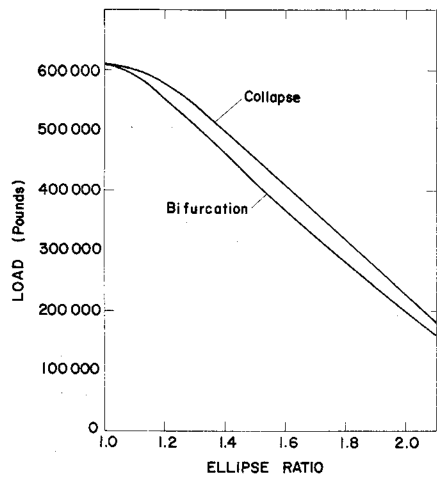

Fig. 2 Collapse and bifurcation buckling loads for elliptic cones.

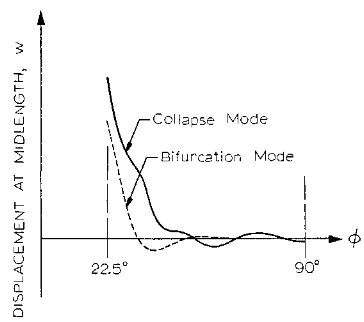

Fig. 5 Buckling and collapse modes for cylinder with cutout.

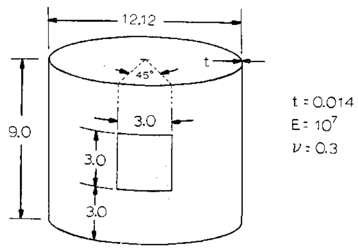

Fig. 3 Cylindrical shell with cutout.

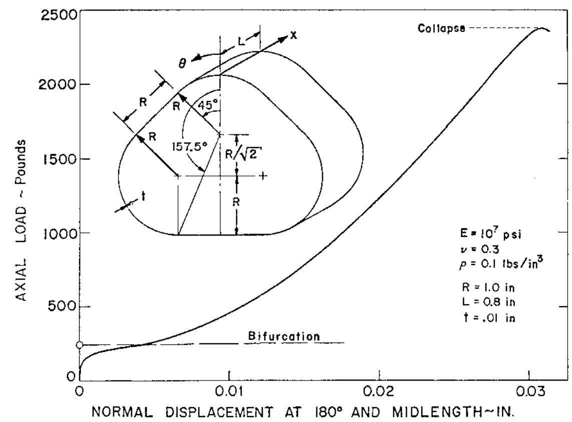

Fig. 6 Pear-shaped cylinder normal displacement at 180° and midlength.

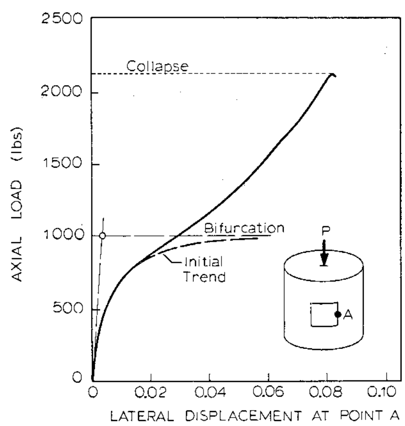

Fig. 4 Cylinder with unreinforced cutout.

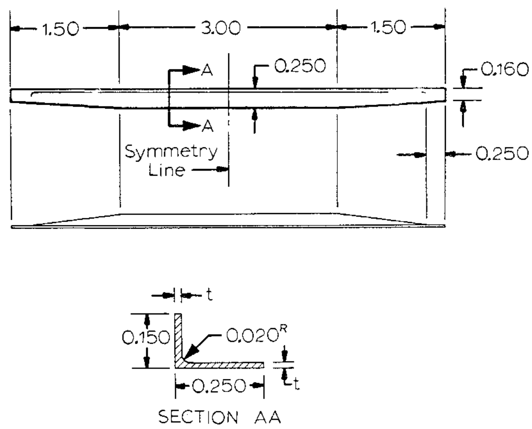

Fig. 7 Stiffener geometry.

somewhat higher than the bifurcation buckling load; this may be expected because the nonlinear analysis takes into account that with increasing load a relatively larger portion of the load is carried in the part of the shell with sharper curvature. Figure 2 shows how the collapse load as well as the bifurcation buckling load vary with the ellipse ratio. It appears that over the entire range the bifurcation load gives a fair approximation.

The effect of static indeterminance is more pronounced in the next example. A cylindrical shell with two diametrically opposite cutouts (Fig. 3) is loaded through uniform end shortening. Figure 4 shows the difference between nonlinear behavior and "bifurcation behavior." Buckling of the free edge at the cutout occurs according to the classical theory at a 1000 lb load. Actually this free edge starts to bend immediately at application of load. There seems to be a tendency of the lateral displacement to approach infinity at this load level as indicated by the broken curve. However, as the area around the cutout weakens, the stresses are redistributed so that more load is carried in the parts of the shell away from the cutouts, and collapse does not occur until a considerably higher load is applied.

A collapse mode can be defined as the displacement mode, the growth of which is dominant at the maximum point in the load displacement curve. It is obtained as the difference in displacements between two neighboring solutions. The collapse mode is compared to the buckling mode (bifurcation) in Fig. 5. It is seen that the buckling mode is confined to the area of the cutout, but the collapse mode extends over the entire shell surface.

In Fig. 6, an even more drastic example is demonstrated. For the cylinder composed of flat plates and circular cylindrical panels, the bifurcation load indicates approximately the load level at which the flat plates would buckle. The maximum in the load displacement curve indicates the load level at which the curved elements collapse. It can be seen that the collapse load is more than ten times higher than the bifurcation buckling load.

The aforementioned circular cylinder with two 45° cutouts also was analyzed after reinforcing stringers were added at the edges of the cutout. Figure 7 shows the geometry of the reinforcement. With this reinforcement, the stresses are rather uniformly distributed over the shell surface, and buckling occurs in an area away from the cutout. The bifurcation buckling load in this case

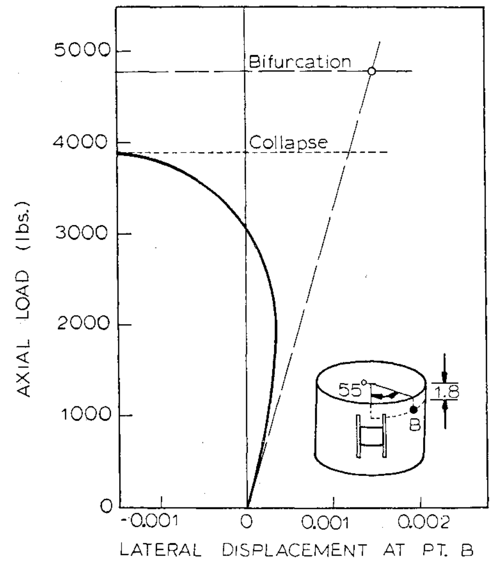

Fig. 8 Cylinder with reinforced cutout.

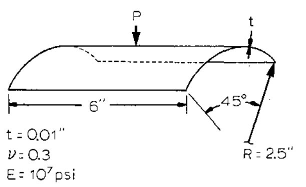

Fig. 9 Cylindrical panel.

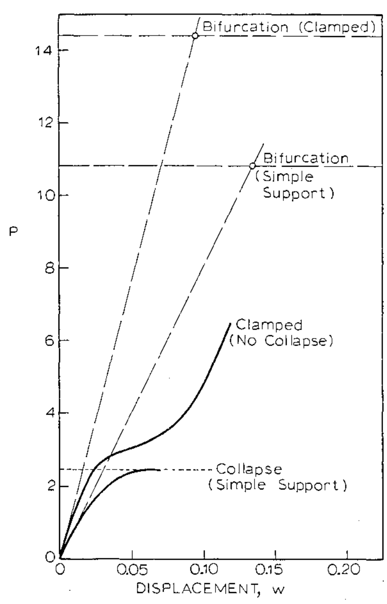

Fig. 10 Bending of cylindrical panel.

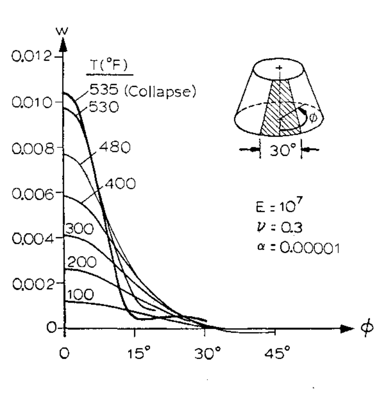

Fig. 11 Displacement for cone heated on strip.

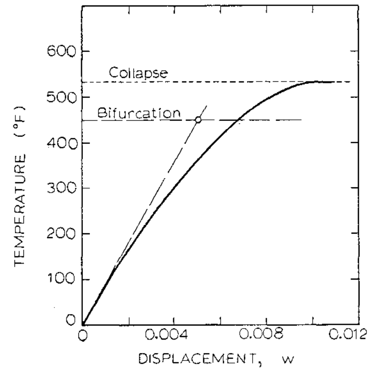

Fig. 12 Load displacement curve for heated cone.

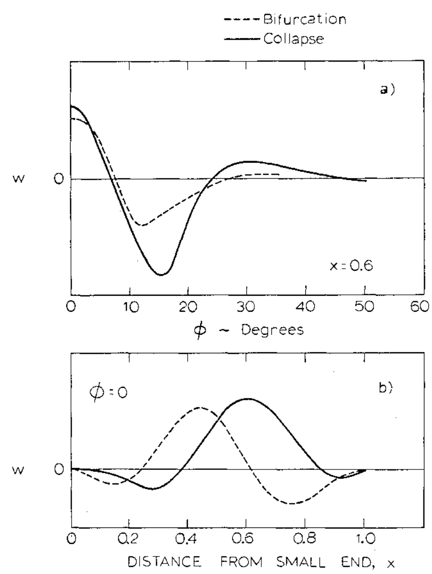

Fig. 13 Collapse and buckling modes for heated cone.

is somewhat higher than the collapse load. This condition may exist because the bifurcation theory does not account for the degrading modifications in geometry caused by (nonlinear) local deformations. The maximum displacement in the collapse mode occurs at a point about 30° away from the cutout edge and approximately halfway between shell edge and cutout edge. Figure 8 shows the lateral displacement at this point as a function of the applied load.

A more drastic example of this type is afforded by the bending of a cylindrical panel with free longitudinal edges (Fig. 9). The effect is similar to that of bending of complete cylinders as discussed by Brazier.[5] The distortion of the cross section reduces the bending stiffness of the shell and collapse occurs as shown in Fig. 10 at a load which for the simply supported shell is less than one quarter of the critical load predicted by classical buckling theory. However, if the edges are restrained from motion in the axial direction, a tension load will develop in the axial direction and for the clamped shell the load increases monotonically with the displacement; collapse will not occur at all.

In the next example thermal buckling is considered. The circular conical shell shown in Fig. 11 is subjected to elevated temperature over two diagonally opposite strips, each covering 30° of circumference while the remaining part of the surface remains at base temperature. The figure shows displacements at midlength as a function of the circumferential coordinate and for different temperatures. The maximum displacement (at $\phi = 0$) is shown in Fig. 12 as a function of the temperature. It is seen that collapse occurs at a temperature of 535°F. The bifurcation buckling analysis indicates that the critical temperature is 450°F. Buckling and collapse modes are compared in Fig. 13.

Conclusions

A branch for bifurcation analysis was added to the STAGS computer program for nonlinear analysis of general shells. In the general case, the bifurcation theory does not represent a rigorous solution of the physical problem. Therefore, a numerical study was undertaken to shed some light on the usefulness of the linearized bifurcation buckling theory. From this analysis, it can be concluded that if the shell structure allows considerable redistribution of stresses the bifurcation buckling load may be only a small fraction of the collapse load. In the similar case of a plate with inplane loading and supported edges, it is well known that the buckling load can be very small in comparison to the collapse load. However, in this case, the bifurcation buckling load has physical significance as it defines the load level at which lateral displacements begin to develop. For a more complicated structure, the lateral displacements occur immediately with the onset of loading and the classical buckling load may not have any significance as a design parameter.

Another reason for discrepancy between bifurcation buckling and collapse loads is that at loads well below predicted bifurcation, considerable distortion of the geometry occurs. It is known that, for instance, a curved tube will flatten with application of a bending moment. Any computer program for bifurcation buckling of shells with general shape or loading should be used with some caution. The bifurcation buckling load can be assumed to approximate the failure load only if such an assumption is justified by analysis of similar cases. For other cases, the bifurcation buckling analysis may still be useful in preparation for a complete nonlinear analysis, particularly to aid in the choice of a grid and of initial load step.

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